ДРУГИ ПИСМЕНИ ЗАДАТАК

1) Нека су вектори $\vec{a}$ и $\vec{b}$ такви да је:

(I) $|\vec{a}| = 1$ $|\vec{b}| = 4$ $\measuredangle(\vec{a}, \vec{b}) = \frac{\pi}{6}$ — ОДРЕДИТИ: (I) $\measuredangle(2\vec{a}-\vec{b}, 2\vec{a}+\vec{b})$

(II) $|\vec{a}| = 2$ $|\vec{b}| = 6$ $\measuredangle(\vec{a}, \vec{b}) = \frac{\pi}{3}$ (II) $\measuredangle(\vec{a}-\frac{1}{2}\vec{b}, \vec{a}+\frac{1}{2}\vec{b})$

(I) $\cos\measuredangle(2\vec{a}-\vec{b}, 2\vec{a}+\vec{b}) = \dfrac{(2\vec{a}-\vec{b})(2\vec{a}+\vec{b})}{|2\vec{a}-\vec{b}|\cdot|2\vec{a}+\vec{b}|}$

$(2\vec{a}-\vec{b})\cdot(2\vec{a}+\vec{b}) = 4|\vec{a}|^2 - |\vec{b}|^2 = 4-16 = -12$

$|2\vec{a}-\vec{b}|^2 = (2\vec{a}-\vec{b})\cdot(2\vec{a}-\vec{b}) = 4|\vec{a}|^2 - 2\cdot 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 = 4 - 4\cdot|\vec{a}|\cdot|\vec{b}|\cdot\cos\measuredangle(\vec{a},\vec{b}) + 16$

$= 20 - 4\cdot 4\cdot\cos\frac{\pi}{6} = 20 - 16\cdot\frac{\sqrt{3}}{2} = 20 - 8\sqrt{3} = 4(5-2\sqrt{3})$

$|2\vec{a}+\vec{b}|^2 = (2\vec{a}+\vec{b})\cdot(2\vec{a}+\vec{b}) = 4\cdot|\vec{a}|^2 + 2\cdot 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 = 4 + 4\cdot|\vec{a}|\cdot|\vec{b}|\cdot\cos\frac{\pi}{6} + 16$

$= 20 + 16\cdot\frac{\sqrt{3}}{2} = 20 + 8\sqrt{3} = 4(5+2\sqrt{3})$.

$\cos\measuredangle(2\vec{a}-\vec{b}, 2\vec{a}+\vec{b}) = \dfrac{-12}{4\sqrt{(5-2\sqrt{3})(5+2\sqrt{3})}} = -\dfrac{3}{\sqrt{13}} = -\dfrac{3\sqrt{13}}{13}$.

(II) $\cos\measuredangle(\vec{a}-\frac{1}{2}\vec{b}, \vec{a}+\frac{1}{2}\vec{b}) = \dfrac{(\vec{a}-\frac{1}{2}\vec{b})(\vec{a}+\frac{1}{2}\vec{b})}{|\vec{a}-\frac{1}{2}\vec{b}|\cdot|\vec{a}+\frac{1}{2}\vec{b}|}$

$(\vec{a}-\frac{1}{2}\vec{b})\cdot(\vec{a}+\frac{1}{2}\vec{b}) = |\vec{a}|^2 - \frac{1}{4}|\vec{b}|^2 = 4 - \frac{1}{4}\cdot 36 = 4-9 = -5$

$|\vec{a}-\frac{1}{2}\vec{b}|^2 = (\vec{a}-\frac{1}{2}\vec{b})(\vec{a}-\frac{1}{2}\vec{b}) = |\vec{a}|^2 - \vec{a}\cdot\vec{b} + \frac{1}{4}|\vec{b}|^2 = 4 - |\vec{a}||\vec{b}|\cdot\cos\measuredangle(\vec{a},\vec{b}) + \frac{1}{4}\cdot 36 =$

$= 13 - 12\cdot\cos\frac{\pi}{3} = 13 - 6 = 7$

$|\vec{a}+\frac{1}{2}\vec{b}|^2 = (\vec{a}+\frac{1}{2}\vec{b})(\vec{a}+\frac{1}{2}\vec{b}) = |\vec{a}|^2 + \vec{a}\cdot\vec{b} + \frac{1}{4}|\vec{b}|^2 = 13 + 12\cdot\frac{1}{2} = 19$.

$\cos\measuredangle(\vec{a}-\frac{1}{2}\vec{b})(\vec{a}+\frac{1}{2}\vec{b}) = \dfrac{-5}{\sqrt{7}\cdot\sqrt{19}} = -\dfrac{5}{\sqrt{133}} = -\dfrac{5\sqrt{133}}{133}$.

2) (I) Ако је $|\vec{a}| = 10$ $|\vec{b}| = 2$ и $|\vec{a}\times\vec{b}| = \frac{100}{13}$, наћи $\vec{a}\cdot\vec{b}$.

$|\vec{a}\times\vec{b}| = |\vec{a}|\cdot|\vec{b}|\cdot\sin\measuredangle(\vec{a},\vec{b})$;

$\frac{100}{13} = 20\cdot\sin\measuredangle(\vec{a},\vec{b}) \Rightarrow \sin\measuredangle(\vec{a},\vec{b}) = \frac{5}{13}$

$\sin^2\measuredangle(\vec{a},\vec{b}) + \cos^2\measuredangle(\vec{a},\vec{b}) = 1 \Rightarrow \cos\measuredangle(\vec{a},\vec{b}) = \frac{12}{13}$

$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\measuredangle(\vec{a},\vec{b}) = 10\cdot 2\cdot\frac{12}{13} = \frac{240}{13}$.

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2) Ако је $|\vec{a}|=10$, $|\vec{b}|=2$ и $\vec{a}\cdot\vec{b}=12$, наћи $|\vec{a}\times\vec{b}|$

$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cdot\cos\measuredangle(\vec{a},\vec{b}) \Rightarrow 12=20\cdot\cos\measuredangle(\vec{a},\vec{b}) \Rightarrow \cos\measuredangle(\vec{a},\vec{b})=\frac{3}{5}$

$\sin^2\measuredangle(\vec{a},\vec{b})+\cos^2\measuredangle(\vec{a},\vec{b})=1 \Rightarrow \sin\measuredangle(\vec{a},\vec{b})=\frac{4}{5}$

$|\vec{a}\times\vec{b}|=10\cdot2\cdot\sin\measuredangle(\vec{a},\vec{b})=20\cdot\frac{4}{5}=16.$

3) Израчунати површину троугла ако су дате координате његових темена, а затим и одговарајућу висину.

(I) $A(2,-3,4)$ $B(1,2,-1)$ $C(3,-2,1)$ и висину AD.

(II) $A(1,-2,8)$ $B(0,0,4)$ $C(6,2,0)$ и висину BD.

(I) $\vec{AB}=(-1,5,-5)$, $\vec{AC}=(1,1,-3)$

$$\vec{AB}\times\vec{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 5 & -5\\ 1 & 1 & -3\end{vmatrix}=-10\vec{i}-8\vec{j}-6\vec{k}=(-10,-8,-6)$$

$|\vec{AB}\times\vec{AC}|=\sqrt{100+64+36}=10\sqrt{2}$; $P=\frac{1}{2}|\vec{AB}\times\vec{AC}|=5\sqrt{2}$

$\vec{BC}=(2,-4,2)$; $|\vec{BC}|=\sqrt{4+16+4}=\sqrt{24}=2\sqrt{6}.$

$$P_\Delta=\frac{|\vec{BC}|\cdot|\vec{AD}|}{2} \Rightarrow 5\sqrt{2}=\frac{\cancel{2}\sqrt{6}\cdot|\vec{AD}|}{\cancel{2}} \Rightarrow |\vec{AD}|=\frac{5\sqrt{2}}{\sqrt{6}}=\frac{5\sqrt{12}}{6}=\frac{10\sqrt{3}}{6}=\frac{5\sqrt{3}}{3}$$

(II) $\vec{AB}=(-1,2,-4)$ $\vec{AC}=(5,4,-8)$

$$\vec{AB}\times\vec{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 2 & -4\\ 5 & 4 & -8\end{vmatrix}=0\cdot\vec{i}-28\vec{j}-14\vec{k}=(0,-28,-14)$$

$|\vec{AB}\times\vec{AC}|=\sqrt{4\cdot14^2+14^2}=14\sqrt{5}.$ $P=\frac{1}{2}\cdot14\sqrt{5}=7\sqrt{5}$

$\vec{AC}=(5,4,-8) \Rightarrow |\vec{AC}|=\sqrt{25+16+64}=\sqrt{105}$

$$7\sqrt{5}=\frac{1}{2}\sqrt{105}\cdot|\vec{BD}| \;;\; |\vec{BD}|=\frac{14\sqrt{5}}{\sqrt{105}}=\frac{14\sqrt{5}}{\sqrt{21}\cdot\cancel{\sqrt{5}}}\cdot\frac{\sqrt{21}}{\sqrt{21}}=\frac{14\sqrt{21}}{21}=\frac{2\sqrt{21}}{3}$$

4) Одредити x тако да вектори буду компланарни:

$\vec{a}=(1,x-1,1)$ $\vec{b}=(3,1,2)$ $\vec{c}=(4,4,x-1)$

$$\begin{vmatrix}1 & x-1 & 1\\ 3 & 1 & 2\\ 4 & 4 & x-1\end{vmatrix}=0 \;;\; \begin{vmatrix}-3x+4 & x-1 & -2x+3\\ 0 & 1 & 0\\ -8 & 4 & x-9\end{vmatrix}=0$$

(-3) (-2) + +

$$(-3x+4)(x-9)+8(-2x+3)=0$$

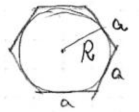
$-3x^2+27x+4x-36-16x+24=0$

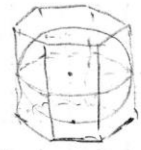
$-3x^2+15x-12=0 \qquad x_{1,2}=\frac{-15\pm\sqrt{225-144}}{-6}=\frac{-15\pm 9}{-6}= \begin{matrix} 1 \\ 4 \end{matrix}$

$x=1$ или $x=4$.

5) (I) Око лопте полупречника R описана је правилна шестострана призма. Израчунати њену површину

$H=2R$





Из нормалног пресека је:

$R=\frac{a\sqrt{3}}{2}$ тј. $a=\frac{2R\sqrt{3}}{3}$

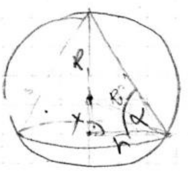
$M=6\cdot a\cdot H=\cancel{6}^{2}\cdot\frac{2R\sqrt{3}}{\cancel{3}}\cdot 2R=8R^2\sqrt{3}$

$B=6\cdot\frac{a^2\sqrt{3}}{4}=\frac{3\sqrt{3}}{2}\cdot\frac{4R^2\cdot\cancel{3}}{\cancel{9}}=2R^2\sqrt{3}.$

$P=2B+M=4R^2\sqrt{3}+8R^2\sqrt{3}=12R^2\sqrt{3}$

(II) У лопти полупречника R уписана је купа. Изводница купе нагнута је према равни основе под углом α. Наћи V купе.

$\operatorname{ctg}\alpha=\frac{r}{R+x}$ тј. $\left.\begin{matrix} r=(R+x)\cdot\operatorname{ctg}\alpha \\ r^2=R^2-x^2 \end{matrix}\right\}$ биће:



$(R+x)^2\operatorname{ctg}^2\alpha=R^2-x^2$ тј. $(R+x)^2\operatorname{ctg}^2\alpha=(R-x)(R+x)$

$(R+x)\operatorname{ctg}^2\alpha=R-x \Rightarrow R\operatorname{ctg}^2\alpha+x\operatorname{ctg}^2\alpha=R-x$

$x(1+\operatorname{ctg}^2\alpha)=R(1-\operatorname{ctg}^2\alpha) \Rightarrow x=\frac{R(1-\operatorname{ctg}^2\alpha)}{\frac{\sin^2\alpha+\cos^2\alpha}{\sin^2\alpha}}=\frac{R(1-\operatorname{ctg}^2\alpha)}{\frac{1}{\sin^2\alpha}}$

$x=R\sin^2\alpha(1-\operatorname{ctg}^2\alpha)$

$H=R+x=R\left(1+\sin^2\alpha\left(\frac{\sin^2\alpha-\cos^2\alpha}{\sin^2\alpha}\right)\right)=R(1+\sin^2\alpha-\cos^2\alpha)$

$\boxed{H=2R\sin^2\alpha}$ $\qquad r=(R+x)\operatorname{ctg}\alpha=2R\sin^2\alpha\cdot\frac{\cos\alpha}{\sin\alpha}=2R\sin\alpha\cos\alpha$

$\boxed{r=R\sin 2\alpha}$

$V=\frac{1}{3}R^2\sin^2 2\alpha\,\pi\cdot 2R\sin^2\alpha=\frac{2R^3\pi}{3}\cdot\sin^2 2\alpha\cdot\sin^2\alpha.$