Израчунати интеграле:

Прва група

IV$_2$

1) $\displaystyle\int_{\pi/4}^{\pi/2} \mathrm{ctg}\, x \, dx$

2) $\displaystyle\int_{1/2}^{1} \arcsin x \, dx$

3) $\displaystyle\int_{0}^{\pi/4} \frac{x \sin x}{\cos^2 x} \, dx$

IV$_1$

1) $\displaystyle\int_{\pi/6}^{\pi} \mathrm{tg}\, x \, dx$

2) $\displaystyle\int_{1}^{\ln 2} (2x+1) e^{x} \, dx$

3) $\displaystyle\int_{1/3}^{\sqrt{3}/3} \frac{dx}{\sqrt{4-9x^2}}$

Друга група

IV$_2$

1) $\displaystyle\int_{0}^{\pi/3} \sin^3 x \, dx$

2) $\displaystyle\int_{0}^{1} \mathrm{arctg}\, x \, dx$

3) $\displaystyle\int_{0}^{3} \sqrt{9-x^2} \, dx$

IV$_1$

1) $\displaystyle\int_{0}^{\pi/6} \cos^3 x \, dx$

2) $\displaystyle\int_{1}^{e^2} x \ln x \, dx$

3) $\displaystyle\int_{1}^{\sqrt[6]{3}} \frac{x^2 \, dx}{1+x^6}$

Одредити површину фигуре ограничене линијама

4) $y = e^{x}$, $y = e^{-x}$, $x = 2$

4) $y = \ln x$, $y = 0$, $x = e$.

(1) У овом задатку требало је решити одређене интеграле методом смене:

I ГРУПА

$$\int_{\pi/4}^{\pi/2} ctg\,x\,dx = \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x}\,dx = \left(\begin{array}{l} \text{смена: } \sin x = t \\ \cos x\,dx = dt \\ x=\frac{\pi}{4}: t=\frac{\sqrt2}{2};\ x=\frac{\pi}{2}: t=1 \end{array}\right) =$$

$$= \int_{\sqrt2/2}^{1} \frac{dt}{t} = \ln|t|\Big|_{\sqrt2/2}^{1} = -\ln\frac{\sqrt2}{2} = \ln\left(\frac{\sqrt2}{2}\right)^{-1} = \ln\left(\frac{1}{\sqrt2}\right)^{-1} = \ln\sqrt2 = \frac{1}{2}\ln 2$$

II ГРУПА

$$\int_0^{\pi/3} \sin^3 x\,dx = \int_0^{\pi/3} \sin x(1-\cos^2 x)\,dx = \int_0^{\pi/3} \sin x\,dx - \int_0^{\pi/3} \sin x\cos^2 x\,dx$$

$$= -\cos x\Big|_0^{\pi/3} - \int_0^{\pi/3} \sin x\cos^2 x\,dx = \left(\begin{array}{l} \text{смена: } \cos x = t \\ -\sin x\,dx = dt \\ x=0: t=1;\ x=\frac{\pi}{3}: t=\frac{1}{2} \end{array}\right) = -\frac{1}{2}+1+\int_1^{1/2} t^2\,dt =$$

$$= \frac{1}{2} - \frac{t^3}{3}\Big|_{1/2}^{1} = \frac{1}{2} - \frac{1}{3}\left(1-\frac{1}{8}\right) = \frac{1}{2} - \frac{7}{24} = \frac{5}{24}$$

Дакле, $I = \frac{5}{24}$.

(2) Задатак се ради парцијалном интеграцијом

I ГРУПА

$$\int_{1/2}^{1} \arcsin x\,dx \quad \left(\begin{array}{l} \text{смена: } u = \arcsin x;\ du = \frac{dx}{\sqrt{1-x^2}} \\ dv = dx \Rightarrow v = x \end{array}\right) =$$

$$= x\cdot\arcsin x\Big|_{1/2}^{1} - \int_{1/2}^{1} \frac{x\,dx}{\sqrt{1-x^2}} = \left(\begin{array}{l} \text{смена } 1-x^2 = t \\ -2x\,dx = dt \\ x=\frac{1}{2}: t=\frac{3}{4};\ x=1: t=0 \end{array}\right) = x\arcsin 1 - \frac{1}{2}\arcsin\frac{1}{2} -$$

$$- \frac{1}{2}\int_1^{3/4} \frac{dt}{t^{1/2}} = \frac{\pi}{2} - \frac{\pi}{12} - \frac{\cancel{1}}{\cancel{2}}\cdot\frac{t^{1/2}}{\cancel{1/2}}\Big|_1^{3/4} = \frac{10\pi}{12} - \frac{\sqrt3}{2} + 1 = \frac{5\pi - 3\sqrt3 + 6}{6}$$

II ГРУПА

$$\int_0^1 arctg\,x\,dx = \left(\begin{array}{l} \text{смена: } u = arctg\,x;\ du = \frac{dx}{1+x^2} \\ dv = dv;\ v = x \end{array}\right) = x\,arctg\,x\Big|_0^1 - \int_0^1 \frac{x\,dx}{1+x^2} =$$

$$= \left(\begin{array}{l} \text{смена: } 1+x^2 = t \\ x\,dx = \frac{dt}{2} \\ x=0: t=1 \\ x=1: t=2 \end{array}\right) = arctg\,1 - \frac{1}{2}\int_1^2 \frac{dt}{t} = \frac{\pi}{4} - \frac{1}{2}\ln|t|\Big|_1^2 = \frac{\pi}{4} - \ln\sqrt2.$$

(3) Комбиновани пример:

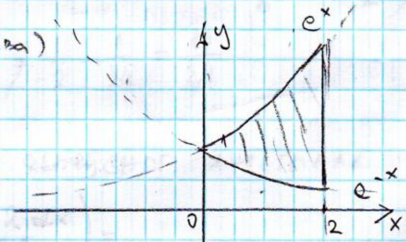
I ГРУПА

$$\int_0^{\pi/4} \frac{x\sin x}{\cos^3 x}\,dx = \left(\begin{array}{l} \text{смена: } u = x;\ du = dx \\ V = \int \frac{\sin x\,dx}{\cos^3 x} = \left(\begin{array}{l} \text{смена: } \cos x = t \\ -\sin x\,dx = dt \end{array}\right) = -\int \frac{dt}{t^3} = -\frac{t^{-2}}{-2} \\ V = \frac{1}{2\cos^2 x} \end{array}\right) =$$

$$= \frac{x}{2\cos^2 x}\Big|_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4} \frac{dx}{\cos^2 x} = \frac{1}{\cancel{2}}\cdot\frac{\pi}{4}\cdot\cancel{2} - \frac{1}{2}tg\,x\Big|_0^{\pi/4} = \frac{\pi}{4} - \frac{1}{2}$$

4. Треба најпре нацртати криве, па израчунати површину фигуре.

I група Криве су $y=e^x$, $y=e^{-x}$ и $x=2$ (права)

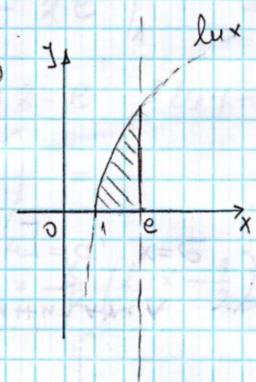


$$P=\int_0^2 (e^x-e^{-x})\,dx = \int_0^2 e^x\,dx - \int_0^2 e^{-x}\,dx$$

$$\left(\begin{array}{l}\text{смене: } -x=t\\ dx=-dt\end{array}\right) = e^x\Big|_0^2 + \int_0^{-2} e^t\,dt =$$

$$= e^2-1-\int_{-2}^{0} e^t\,dt = e^2-1-e^t\Big|_{-2}^{0} = e^2-1-1+e^{-2} = e^2+e^{-2}-2$$

II група Криве су $y=\ln x$, $y=0$, $x=e$ (права)



$$P=\int_1^e \ln x\,dx = \left(\begin{array}{l}\text{смене: } u=\ln x;\ du=\frac{1}{x}\,dx\\ dv=dx;\ v=x\end{array}\right) =$$

$$= x\ln x\Big|_1^e - \int_1^e dx = e - x\Big|_1^e = e-e+1=1.$$