DODATNI ZADACI ZA VEŽBANJE
-Funkcije-

1. Neka su $f:\mathbb{R}\to\mathbb{R}$ i $g:\mathbb{R}\to\mathbb{R}$ realne funkcije. Odrediti: *a)* $f\circ g$ *b)* $g\circ f$

1) $f(x)=x+2\ ,\ g(x)=3x-4$

2) $f(x)=\dfrac{3x+4}{7}\ ,\ g(x)=\dfrac{7x-1}{2}$

3) $f(x)=5x\ ,\ g(x)=4x^2-1$

4) $f(x)=8x^2\ ,\ g(x)=-2x+1$

5) $f(x)=\dfrac{5x-1}{2},\ g(x)=\dfrac{-4x-3}{7}$

Rešenje:

1) $f(x)=x+2\ ,\ g(x)=3x-4$

a) $(f\circ g)(x)=f(g(x))=f(3x-4)=(3x-4)+2=3x-4+2=3x-2$

b) $(g\circ f)(x)=g(f(x))=g(x+2)=3(x+2)-4=3x+6-4=3x+2$

2) $f(x)=\dfrac{3x+4}{7}\ ,\ g(x)=\dfrac{7x-1}{2}$

a) $$(f\circ g)(x)=f(g(x))=f\left(\frac{7x-1}{2}\right)=\frac{3\left(\frac{7x-1}{2}\right)+4}{7}=\frac{\frac{21x-3}{2}+4}{7}=\frac{\frac{21x-3+8}{2}}{7}=\frac{\frac{21x+5}{2}}{7}=\frac{21x+5}{14}$$

b) $$(g\circ f)(x)=g(f(x))=g\left(\frac{3x+4}{7}\right)=\frac{7\left(\frac{3x+4}{7}\right)-1}{2}=\frac{3x+4-1}{2}=\frac{3x-3}{2}$$

3) $f(x)=5x\ ,\ g(x)=4x^2-1$

a) $(f\circ g)(x)=f(g(x))=f(4x^2-1)=5(4x^2-1)=20x^2-5$

b) $(g\circ f)(x)=g(f(x))=g(5x)=4(5x)^2-1=4\cdot 25x^2-1=100x^2-1$

4) $f(x)=8x^2\ ,\ g(x)=-2x+1$

a) $(f\circ g)(x)=f(g(x))=f(-2x+1)=8(-2x+1)^2=8(4x^2-4x+1)=32x^2-32x+8$

b) $(g\circ f)(x)=g(f(x))=g(8x^2)=-2(8x^2)+1=-16x^2+1$

5) $f(x)=\dfrac{5x-1}{2},\ g(x)=\dfrac{-4x-3}{7}$

a) $$(f\circ g)(x)=f(g(x))=f\left(\frac{-4x-3}{7}\right)=\frac{5\left(\frac{-4x-3}{7}\right)-1}{2}=\frac{\frac{-20x-15-7}{7}}{2}=\frac{\frac{-20x-22}{7}}{2}=\frac{-20x-22}{14}=\frac{-10x-11}{7}$$

b) $$(g\circ f)(x)=g(f(x))=g\left(\frac{5x-1}{2}\right)=\frac{-4\left(\frac{5x-1}{2}\right)-3}{7}=\frac{\frac{-20x+4-6}{2}}{7}=\frac{-20x-2}{14}=\frac{-10x-1}{7}$$

2. Neka je $f:\mathbb{R}\to\mathbb{R}$ realna funkcija. Dokazati da je funkcija f bijekcija i odrediti njenu inverznu funkciju.

1) $f(x)=\frac{3x+4}{7}$ 2) $f(x)=\frac{5x-1}{2}$ 3) $f(x)=\frac{-x+11}{3}$ 4) $f(x)=\frac{-10x-3}{13}$

Rešenje:

1) $f(x)=\frac{3x+4}{7}$

injekcija („1-1“)

$$f(x_1)=f(x_2)\Rightarrow\frac{3x_1+4}{7}=\frac{3x_2+4}{7}\quad/\cdot 7$$
$$\Rightarrow 3x_1+4=3x_2+4$$
$$\Rightarrow 3x_1=3x_2\quad/\cdot\tfrac{1}{3}$$
$$\Rightarrow x_1=x_2$$

sirjekcija („na“)

$y_0, x_0=?$

$$f(x_0)=y_o\Leftrightarrow\frac{3x_0+4}{7}=y_0\quad/\cdot 7$$
$$\Leftrightarrow 3x_0+4=7y_0$$
$$\Leftrightarrow 3x_0=7y_0-4\ /\cdot\tfrac{1}{3}$$
$$\Leftrightarrow x_0=\frac{1}{3}(7y_0-4)$$
$$\Leftrightarrow x_0=\frac{7y_0-4}{3}$$

provera:

$$f(x_0)=f\left(\frac{7y_0-4}{3}\right)$$
$$=\frac{3\frac{7y_0-4}{3}+4}{7}$$
$$=\frac{7y_0-4+4}{7}$$
$$=\frac{7y_0}{7}$$
$$=y_0$$

Inverzna funkcija:

$$f^{-1}(f(x))=x$$
$$f^{-1}\left(\frac{3x+4}{7}\right)=x$$
$$t=\frac{3x+4}{7}\Leftrightarrow 7t=3x+4$$
$$\Leftrightarrow 7t-4=3x$$
$$\Leftrightarrow\frac{7t-4}{3}=x$$
$$\boxed{f^{-1}(t)=\frac{7t-4}{3}}$$

2) $f(x)=\frac{5x-1}{2}$

injekcija („1-1“)

$$f(x_1)=f(x_2)\Rightarrow\frac{5x_1-1}{2}=\frac{5x_2-1}{2}\quad/\cdot 2$$
$$\Rightarrow 5x_1-1=5x_2-1$$
$$\Rightarrow 5x_1=5x_2\quad/\cdot\tfrac{1}{5}$$
$$\Rightarrow x_1=x_2$$

sirjekcija („na“)

$y_0, x_0=?$

$$f(x_0)=y_o\Leftrightarrow\frac{5x_0-1}{2}=y_0\quad/\cdot 2$$
$$\Leftrightarrow 5x_0-1=2y_0$$
$$\Leftrightarrow 5x_0=2y_0+1\ /\cdot\tfrac{1}{5}$$
$$\Leftrightarrow x_0=\frac{1}{5}(2y_0+1)$$
$$\Leftrightarrow x_0=\frac{2y_0+1}{5}$$

provera:

$$f(x_0)=f\left(\frac{2y_0+1}{5}\right)$$
$$=\frac{5\frac{2y_0+1}{5}-1}{2}$$
$$=\frac{2y_0+1-1}{2}$$
$$=\frac{2y_0}{2}$$
$$=y_0$$

Inverzna funkcija:

$$f^{-1}(f(x))=x$$
$$f^{-1}\left(\frac{5x-1}{2}\right)=x$$
$$t=\frac{5x-1}{2}\Leftrightarrow 2t=5x-1$$
$$\Leftrightarrow 2t+1=5x$$
$$\Leftrightarrow\frac{2t+1}{5}=x$$
$$\boxed{f^{-1}(t)=\frac{2t+1}{5}}$$

3) $f\left(x\right)=\frac{-x+11}{3}$

injekcija („1-1“)

$$f(x_1)=f(x_2)\Rightarrow\frac{-x_1+11}{3}=\frac{-x_2+11}{3}\quad /\cdot 3$$
$$\Rightarrow -x_1+11=-x_2+11$$
$$\Rightarrow -x_1=-x_2\quad /\cdot(-1)$$
$$\Rightarrow x_1=x_2$$

sirjekcija („na“)

$y_0, x_0=?$

$$f\left(x_0\right)=y_o\Leftrightarrow\frac{-x_0+11}{3}=y_0\quad /\cdot 3$$
$$\Leftrightarrow -x_0+11=3y_0$$
$$\Leftrightarrow -x_0=3y_0-11\ /\cdot\left(-1\right)$$
$$\Leftrightarrow x_0=-3y_0+11$$

provera:

$$f\left(x_0\right)=f\left(-3y_0+11\right)$$
$$=\frac{-\left(-3y_0+11\right)+11}{3}$$
$$=\frac{3y_0-11+11}{3}$$
$$=\frac{3y_0}{3}$$
$$=y_0$$

Funkcija je bijekcija.

Inverzna funkcija:

$$f^{-1}\left(f\left(x\right)\right)=x$$
$$f^{-1}\left(\frac{-x+11}{3}\right)=x$$
$$t=\frac{-x+11}{3}\Leftrightarrow 3t=-x+11$$
$$\Leftrightarrow 3t-11=-x$$
$$\Leftrightarrow -3t+11=x$$
$$\boxed{f^{-1}\left(t\right)=-3t+11}$$

4) $f\left(x\right)=\frac{-10x-3}{13}$

injekcija („1-1“)

$$f(x_1)=f(x_2)\Rightarrow\frac{-10x_1-3}{13}=\frac{-10x_2-3}{13}\quad /\cdot 13$$
$$\Rightarrow -10x_1-3=-10x_2-3$$
$$\Rightarrow -10x_1=-10x_2\quad /\cdot\left(-\tfrac{1}{10}\right)$$
$$\Rightarrow x_1=x_2$$

sirjekcija („na“)

$y_0, x_0=?$

$$f\left(x_0\right)=y_o\Leftrightarrow\frac{-10x_0-3}{13}=y_0\quad /\cdot 13$$
$$\Leftrightarrow -10x_0-3=13y_0$$
$$\Leftrightarrow -10x_0=13y_0+3/\cdot\left(-\tfrac{1}{10}\right)$$
$$\Leftrightarrow x_0=-\frac{1}{10}\left(13y_0+3\right)$$
$$\Leftrightarrow x_0=-\frac{13y_0+3}{10}$$

provera:

$$f\left(x_0\right)=f\left(-\frac{13y_0+3}{10}\right)$$
$$=\frac{-10\left(-\frac{13y_0+3}{10}\right)-3}{13}$$
$$=\frac{13y_0+3-3}{13}$$
$$=\frac{13y_0}{13}$$
$$=y_0$$

Funkcija je bijekcija.

Inverzna funkcija:

$$f^{-1}\left(f\left(x\right)\right)=x$$
$$f^{-1}\left(\frac{-10x-3}{13}\right)=x$$
$$t=\frac{-10x-3}{13}\Leftrightarrow 13t=-10x-3$$
$$\Leftrightarrow 13t+3=-10x$$
$$\Leftrightarrow -\frac{13t+3}{10}=x$$
$$\boxed{f^{-1}\left(t\right)=-\frac{13t+3}{10}}$$